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A note on the almost sure limit theorem for the product of partial sums

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Abstract

We present an almost sure limit theorem for the product of the partial sums of i.i.d. positive random variables. We also prove a corresponding almost sure limit theorem for a triangular array.

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1. Introduction and main results

It is well known that the products of independent, identically distributed (i.i.d.), positive, square integrable random variables (r.v.'s) are asymptotically lognormal. This fact is an immediate consequence of the classical central limit theorem (CLT) and is sometimes referred to as the "geometric average" CLT as opposed to the classical "arithmetic average" CLT. In the present paper, we are interested in the limiting law of the products of sums or the "geometric average of arithmetic averages" CLT for i.i.d. variables. It appears that their asymptotic behavior is rather similar. This point was first argued by Arnold and Villaseñor [1], who considered the limiting properties of the sums of records. In their paper Arnold and Villaseñor obtained the following version of the CLT for a sequence (Y_n) of i.i.d. exponential r.v.'s with the mean equal to one:

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$$\frac{\sum_{k=1}^{n}\log(S_k)-n\log(n)+n}{\sqrt{2n}}\xrightarrow{D}\mathcal{N}$$

as $n\to\infty$, where $S_k=Y_1+\cdots+Y_k$, $k=1,2,\ldots$, and $\mathcal{N}$ is a standard normal r.v. Their proof was heavily based on a very special property of exponential (gamma) distributions, namely the independence of ratios of subsequent partial sums and the last sum. It used also Resnick's [2] result on weak limits of records. Rempała and Wesołowski [3] have noted that this limit behavior of a product of partial sums has a universal character and holds for any sequence of square integrable, positive i.i.d. random variables. Namely, they have proved the following.

Theorem 1. *Let* (Y_n) *be a sequence of i.i.d. positive square integrable random variables with* $\mathbf{E}(Y_1)=\mu$, $\mathbf{Var}(Y_1)=\sigma^2>0$ *and the coefficient of variation* $\gamma=\sigma/\mu$. *Then*

$$\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{1/(\gamma\sqrt{n})}\xrightarrow{D}\mathrm{e}^{\sqrt{2}\mathcal{N}}.$$

The above result was swiftly extended by Qi [4], who has shown that whenever (Y_n) is in the domain of attraction of a stable law $\mathcal{L}$ with index $\alpha\in(1,2]$ then there exists a numerical sequence A_n (which for $\alpha=2$ can be taken as $\sigma\sqrt{n}$) such that

$$\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\mu/A_n}\xrightarrow{D}\mathrm{e}^{((\Gamma(\alpha+1))^{1/\alpha})\mathcal{L}},$$

where $\Gamma(\alpha+1)=\int_0^\infty x^\alpha\,\mathrm{e}^{-x}\mathrm{d}x$.

The purpose of the current note is to obtain an almost sure version of Theorem 1. It is well known that for i.i.d. r.v.'s the almost sure central limit theorem (ASCLT) holds under the same assumptions as the CLT, but in general, the mere existence of the weak limit does not always imply the almost sure limiting result for the logarithmic averages and thus the investigation of a general almost sure limit theorem seems somewhat more challenging. We refer to the survey papers [5] and [6] or [7] for more details.

It turns out, however, that the ASCLT holds for partials sums of i.i.d. r.v.'s in the setting of Theorem 1. Our main result in this note is the following.

Theorem 2. *Let* (Y_n) *be a sequence of i.i.d. positive random variables with* $\mathbf{E}(Y_1)=\mu>0$ *and* $\mathbf{Var}(Y_1)=\sigma^2$. *Denote* $\gamma=\sigma/\mu$ *the coefficient of variation. Then for any real* x,

$$\lim_{N\to\infty}\frac{1}{\log N}\sum_{n=1}^{N}\frac{1}{n}\mathbf{I}\left(\left(\frac{\prod_{k=1}^{n}S_k}{n!\mu^n}\right)^{\frac{1}{\gamma\sqrt{n}}}\le x\right)=F(x)\quad a.s. \tag{1.1}$$

where $F(\cdot)$ *is the distribution function of the random variable* $\mathrm{e}^{\sqrt{2}\mathcal{N}}$.

In order to prove the above theorem we first establish the ASCLT for certain triangular arrays of random variables. In the sequel we shall use the following notation. Let $b_{k,n} = \sum_{i=k}^{n} 1/i$ and $s_{k,n} = \left(\sum_{i=1}^{k} b_{i,n}^2\right)^{1/2}$ for $k \le n$ with $b_{k,n} = 0$ if $k > n$. Furthermore, let $Z_1, Z_2, \ldots$ be a sequence of i.i.d. random variables with zero mean and unit variance. We define a triangular array $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ as $X_{k,n} = b_{k,n} Z_k$ and set $S_{k,n} = X_{1,n} + X_{2,n} + \cdots + X_{k,n}$ for $1 \le k \le n$.

In this setting we establish an almost sure central limit theorem for the triangular array $(X_{k,n})$.

Theorem 3. *Let $(X_{k,n})$ be the triangular array defined above. Then the following almost sure limit theorem holds:*

$$\forall_x \lim_{n\to\infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{I}(S_{k,k}/s_{k,k} \le x) = \Phi(x) \quad a.s. \tag{1.2}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

2. Proofs

2.1. Auxiliary results

The following two lemmas will be needed in course of the proof of Theorems 2 and 3. The first one provides some alternative forms of calculating the quantities $s_{n,n}$.

Lemma 1. $s_{n,n}^2 = 2n - b_{1,n}$.

Proof. Observe that by the definition of $s_{n,n}$

$$s_{n,n}^2 = \sum_{i=1}^{n} b_{i,n}^2 = b_{1,n} + 2 \sum_{k=2}^{n} \sum_{i=1}^{k-1} \frac{1}{k} = b_{1,n} + 2 \sum_{k=2}^{n} \frac{k-1}{k} = 2n - b_{1,n}. \quad \square$$

In the sequel we write "$\ll$" for the inequality "$\le$" up to some universal constant.

Lemma 2. *Let $\xi_1, \xi_2, \ldots$ be a sequence of zero mean, uniformly bounded random variables. Assume that $\mathbf{E}(\xi_k \xi_l) \ll (k/l)^{\varepsilon}$ for any $1 \le k < l$ and some $\varepsilon > 0$. Then*

$$\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \xi_k \to 0 \quad a.s.$$

Proof. Let us set

$$\mu_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \xi_k.$$

Note that

$$\mathbf{E}\left(\sum_{k=1}^{n} \frac{1}{k} \xi_k\right)^2 = \sum_{k=1}^{n} \frac{1}{k^2} \mathbf{E}\xi_k^2 + 2 \sum_{1 \le k < l \le n} \frac{\mathbf{E}(\xi_k \xi_l)}{kl}.$$

By our assumptions, the first term above is bounded. As to the second term, we have

$$\sum_{1\le k<l\le n}\frac{|\mathbf{E}(\xi_k\xi_l)|}{kl}\le\sum_{1\le k<l\le n}\frac{1}{kl}\left(\frac{k}{l}\right)^{\varepsilon}\ll\sum_{l=1}^{n}\frac{1}{l^{1+\varepsilon}}\sum_{k=1}^{l}\frac{1}{k^{1-\varepsilon}}\ll\sum_{l=1}^{n}\frac{1}{l}\ll\log n,$$

which implies

$$\mathbf{E}\mu_n^2\ll(\log n)^{-1}.$$

Now, defining $n_k=\exp(k^{\delta})$ for some $\delta>1$, we obtain

$$\mathbf{E}\mu_{n_k}^2\ll k^{-\delta},$$

which entails

$$\sum_{k=1}^{\infty}\mathbf{E}\mu_{n_k}^2<\infty,$$

and via Borel–Cantelli lemma we have

$$\mu_{n_k}\to 0\quad\text{a.s.}$$

Notice that

$$\frac{\log n_{k+1}}{\log n_k}=\frac{(1+k)^{\delta}}{k^{\delta}}\to 1\quad\text{as}\quad k\to\infty.$$

Let k be such that $n_k<n\le n_{k+1}$, then

$$\begin{aligned}|\mu_n|&\le\frac{1}{\log n}\sum_{i=1}^{n}\frac{1}{i}|\xi_i|\le\frac{1}{\log n_k}\sum_{i=1}^{n_k}\frac{1}{i}|\xi_i|+\frac{1}{\log n_k}\sum_{i=n_k}^{n_{k+1}}\frac{1}{i}|\xi_i|\\&\ll|\mu_{n_k}|+\frac{1}{\log n_k}(\log n_{k+1}-\log n_k)\ll|\mu_{n_k}|+\left(\frac{\log n_{k+1}}{\log n_k}-1\right),\end{aligned}$$

which implies

$$\lim_{n\to\infty}\mu_n=0\quad\text{a.s.}\quad\square$$

2.2. Proof of Theorem 3

Let $\varepsilon>0$; observe that

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{s_{n,n}^2}\sum_{k=1}^{n}\mathbf{E}(X_{k,n}^2\mathbf{I}(|X_{k,n}|\ge\epsilon s_{n,n}))&=\lim_{n\to\infty}\frac{1}{s_{n,n}^2}\sum_{k=1}^{n}\mathbf{E}\left(b_{k,n}^2Z_k^2\mathbf{I}(|b_{k,n}Z_k|\ge\epsilon\sqrt{2n})\right)\\&=\lim_{n\to\infty}\frac{1}{s_{n,n}^2}\sum_{k=1}^{n}b_{k,n}^2\mathbf{E}\left(Z_k^2\mathbf{I}\left(|Z_k|\ge\frac{\epsilon\sqrt{2n}}{b_{k,n}}\right)\right)\\&=\lim_{n\to\infty}\frac{1}{s_{n,n}^2}\sum_{k=1}^{n}b_{k,n}^2\mathbf{E}\left(Z_1^2\mathbf{I}\left(|Z_1|\ge\frac{\epsilon\sqrt{2n}}{b_{k,n}}\right)\right)\\&\le\lim_{n\to\infty}\mathbf{E}\left(Z_1^2\mathbf{I}\left(|Z_1|\ge\frac{\epsilon\sqrt{2n}}{\log(n)}\right)\right)=0,\end{aligned}$$

and hence we obtain the Lindeberg condition for the triangular array $(X_{k,n})$. Thus we have the central limit theorem:

$$S_{n,n}/s_{n,n} \xrightarrow{\mathcal{D}} \mathcal{N} \quad \text{as} \quad n \to \infty,$$

which is equivalent to

$$\mathbf{E} f(S_{n,n}/s_{n,n}) \to \mathbf{E} f(\mathcal{N}) \quad \text{as} \quad n \to \infty$$

for any bounded Lipschitz-continuous function f. In view of the above, in order to establish the validity of (1.2) it suffices to show (see, [8])

$$\mu_n = \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left(f(S_{k,k}/s_{k,k}) - \mathbf{E} f(S_{k,k}/s_{k,k}) \right) \to 0 \quad \text{a.s.}$$

Note that, for $l > k$,

$$\begin{aligned} S_{l,l} - S_{k,k} &= (b_{1l} Z_1 + b_{2l} Z_2 + \cdots + b_{ll} Z_l) - (b_{1k} Z_1 + b_{2k} Z_2 + \cdots + b_{kk} Z_k) \\ &= b_{k+1,l} S_k + (b_{k+1,l} Z_{k+1} + \cdots + b_{ll} Z_l). \end{aligned}$$

Denoting $Q_k = f(S_{k,k}/s_{k,k}) - \mathbf{E} f(S_{k,k}/s_{k,k})$, for $l > k$ we have

$$\begin{aligned} |\mathbf{E}(Q_k Q_l)| &= \left| \mathbf{Cov}\left(f\left(\frac{S_{k,k}}{s_{k,k}}\right), f\left(\frac{S_{l,l}}{s_{l,l}}\right) \right) \right| \\ &\le \left| \mathbf{Cov}\left(f\left(\frac{S_{k,k}}{s_{k,k}}\right), f\left(\frac{S_{l,l}}{s_{l,l}}\right) - f\left(\frac{S_{l,l} - S_{k,k} - b_{k+1,l} S_k}{s_{l,l}}\right) \right) \right| \\ &\quad + \left| \mathbf{Cov}\left(f\left(\frac{S_{k,k}}{s_{k,k}}\right), f\left(\frac{S_{l,l} - S_{k,k} - b_{k+1,l} S_k}{s_{l,l}}\right) \right) \right|. \end{aligned}$$

Since $S_{l,l} - S_{k,k} - b_{k+1,n} S_k$ and $S_{k,k}$ are independent,

$$\mathbf{Cov}\left(f\left(\frac{S_{k,k}}{s_{k,k}}\right), f\left(\frac{S_{l,l} - S_{k,k} - b_{k+1,l} S_k}{s_{l,l}}\right) \right) = 0.$$

By the Lipschitz property of f and the Jensen inequality we have

$$\begin{aligned} &\left| \mathbf{Cov}\left(f\left(\frac{S_{k,k}}{s_{k,k}}\right), f\left(\frac{S_{l,l}}{s_{l,l}}\right) - f\left(\frac{S_{l,l} - S_{k,k} - b_{k+1,l} S_k}{s_{l,l}}\right) \right) \right| \ll \mathbf{E}\left(\frac{|S_{k,k} + b_{k+1,l} S_k|}{s_{l,l}} \right) \\ &\quad \ll \mathbf{E}\left(\frac{|S_{k,k}|}{s_{l,l}} \right) + \mathbf{E}\left(\frac{|b_{k+1,l} S_k|}{s_{l,l}} \right) \ll \frac{s_{k,k}}{s_{l,l}} + b_{k+1,l} \frac{\sqrt{k}}{s_{l,l}} \ll \frac{\sqrt{k}}{\sqrt{l}} (1 + \log(l/k)) \ll \left(\frac{k}{l} \right)^{\epsilon} \end{aligned}$$

for some $0 < \epsilon < 1/2$, which entails

$$|\mathbf{E}(Q_k Q_l)| \ll \left(\frac{k}{l} \right)^{\epsilon},$$

and thus by Lemma 2 we obtain (1.2). Finally, we are in position to prove our main result. □

2.3. *Proof of Theorem 2*

Let $C_k = S_k/(\mu k)$. Observing

$$\frac{1}{\gamma \sqrt{2n}} \sum_{k=1}^{n} (C_k - 1) = \frac{1}{\gamma \sqrt{2n}} \sum_{k=1}^{n} \left(\frac{S_k}{\mu k} - 1 \right) = \frac{1}{\sqrt{2n}} \sum_{k=1}^{n} X_{k,n} = \frac{1}{\sqrt{2n}} S_{n,n},$$

we see that (1.2) is equivalent to

$$\forall_x \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{I}\left(\frac{1}{\gamma\sqrt{2n}} \sum_{k=1}^{n} (C_k - 1) \le x\right) = \Phi(x) \quad \text{a.s.} \tag{2.1}$$

Note that in order to prove (1.1) it is sufficient to show that

$$\forall_x \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \mathbf{I}\left(\frac{1}{\gamma\sqrt{2n}} \sum_{k=1}^{n} \log C_k \le x\right) = \Phi(x) \quad \text{a.s.} \tag{2.2}$$

To this end note that by the law of the iterated logarithm we have for $k \to \infty$

$$|C_k - 1| = O((\log(\log k)/k)^{1/2}) \quad \text{a.s.}$$

Since for $|x| < 1$ we have $\log(1+x) = x - R(x)$ with $\lim_{x\to 0} R(x)/x^2 = 1/2$, thus

$$\left|\sum_{k=1}^{n} \log(C_k) - \sum_{k=1}^{n} (C_k - 1)\right| \ll \sum_{k=1}^{n} (C_k - 1)^2 \ll \sum_{k=1}^{n} \log(\log k)/k \ll \log(\log n)\log(n) \quad \text{a.s.}$$

Hence for almost every event ω and any $\varepsilon > 0$ there exists $n_0 = n_0(\omega, \varepsilon, x)$ such that for $n > n_0$

$$\begin{aligned} \mathbf{I}\left(\frac{1}{\gamma\sqrt{2n}} \sum_{k=1}^{n} (C_k - 1) \le x - \varepsilon\right) &\le \mathbf{I}\left(\frac{1}{\gamma\sqrt{2n}} \sum_{k=1}^{n} \log C_k \le x\right) \\ &\le \mathbf{I}\left(\frac{1}{\gamma\sqrt{2n}} \sum_{k=1}^{n} (C_k - 1) \le x + \varepsilon\right) \end{aligned}$$

and thus (2.1) implies (2.2). □

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